

# Worksheet 1

## Problems using Vieta's formulas

1. First, we rewrite  $x_1^2 + x_2^2$  in terms of elementary symmetric polynomials:  
 $x_1^2 + x_2^2 = x_1^2 + 2x_1x_2 + x_2^2 - 2x_1x_2 = (x_1 + x_2)^2 - 2x_1x_2$ . By Vieta's formulas we have  $x_1 + x_2 = -5$  and  $x_1x_2 = -3$ . We substitute in the expression in order to get  $x_1^2 + x_2^2 = (-5)^2 - 2(-3) = 25 + 6 = 31$ .
2. First, we rewrite  $x_1^2 + x_2^2$  in terms of elementary symmetric polynomials:  
 $x_1^2 + x_2^2 = x_1^2 + 2x_1x_2 + x_2^2 - 2x_1x_2 = (x_1 + x_2)^2 - 2x_1x_2$ . By Vieta's formulas we have  $x_1 + x_2 = -11$  and  $x_1x_2 = 12$ . We substitute in the expression in order to get  $x_1^2 + x_2^2 = (-11)^2 - 2 \cdot 12 = 121 - 24 = 97$ .
3.  $\frac{1}{x_1} + \frac{1}{x_2} = \frac{x_2}{x_1x_2} + \frac{x_1}{x_1x_2} = \frac{x_1+x_2}{x_1x_2}$ . From Vieta's formulas, we have  $x_1 + x_2 = -9$ ,  $x_1x_2 = 33$ . We substitute and get:  $\frac{x_1+x_2}{x_1x_2} = \frac{-9}{33} = -\frac{3}{11}$ .
4. We first rewrite the desired expression in terms of the elementary symmetric polynomials.  
 $(x_1^3 + x_2^3) + (x_1^2 + x_2^2) + (x_1 + x_2) = (x_1 + x_2)(x_1^2 + x_2^2 - x_1x_2) + (x_1^2 + 2x_1x_2 + x_2^2 - 2x_1x_2) + (x_1 + x_2) = (x_1 + x_2)((x_1 + x_2)^2 - 3x_1x_2) + (x_1 + x_2)^2 + (x_1 + x_2) - 2x_1x_2$   
From Vieta's formulas, we know that  $x_1 + x_2 = 8$  and  $x_1x_2 = 11$ . We substitute:  
 $8(8^2 - 3 \cdot 11) + 8^2 + 8 - 2 \cdot 11 = 8(64 - 33) + 64 + 8 - 22 = 8 \cdot 31 + 50 = 8 \cdot 31 + 8 \cdot 6 + 2 = 8 \cdot 37 + 2 = 298$
5. First, we will use the following identity:  $|a| = \sqrt{a^2}$ . This leads us to  
 $|x_1 - x_2| = \sqrt{(x_1 - x_2)^2} = \sqrt{x_1^2 - 2x_1x_2 + x_2^2} = \sqrt{x_1^2 + 2x_1x_2 + x_2^2 - 4x_1x_2} = \sqrt{(x_1 + x_2)^2 - 4x_1x_2}$ .  
Then we use Vieta's formulas to get  $x_1 + x_2 = 15$ ,  $x_1x_2 = 36$ . The result is  
 $\sqrt{15^2 - 4 \cdot 36} = \sqrt{225 - 144} = \sqrt{81} = 9$
6.  $x_1 - x_1^2 + x_2 - x_2^2 = x_1 + x_2 - (x_1^2 + x_2^2) = x_1 + x_2 - (x_1^2 + 2x_1x_2 + x_2^2 - 2x_1x_2) = x_1 + x_2 - ((x_1 + x_2)^2 - 2x_1x_2)$   
From Vieta's formulas we know that  $x_1 + x_2 = 12$  and  $x_1x_2 = 19$ . We substitute and get  $12(1 - 12) + 2 \cdot 19 = -12$ .
7. We remove the parentheses and express in terms of the elementary symmetric polynomials:  
 $x_1^2 - 2x_1 \frac{1}{x_1} + \frac{1}{x_1^2} + x_2^2 - 2x_2 \frac{1}{x_2} + \frac{1}{x_2^2} = x_1^2 + x_2^2 + \frac{1}{x_1^2} + \frac{1}{x_2^2} - 4 = x_1^2 + x_2^2 + \frac{x_1^2 + x_2^2}{(x_1x_2)^2} - 4 = (x_1^2 + x_2^2)(1 + \frac{1}{(x_1x_2)^2}) - 4$ .  
By Vieta's formulas we have  $x_1 + x_2 = 4$  and  $x_1x_2 = 1$ . We substitute and get:  $(4^2 - 2)(1 + 1) - 4 = (16 - 2) \cdot 2 - 4 = 24$
8. From Vieta's formulas we have  $x_1x_2 = a^2 - 2a + 1 = (a - 1)^2$ . Since it is a perfect square,  $(a - 1)^2 \geq 0$ , with equality for  $a = 1$ . So the answer is  $a = 1$  and the minimal value is 0.